ON THE INEQUALITIES FOR MOMENTS OF BRANCHING RANDOM PROCESSES

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Abstract: *estimations for moments of random variables play an important role in the probability theory. Many results were obtains for the moments of the sum of independent random variables(see [4],[6]). However, the estimates for moments of processes of the form (0.1) have been little studied. It is well known that moments* $EX_n^m, m \geq 1$*, are obtained by* m *times differentiating the generating function* Es^{X_n}*. However, the expression for* m^{th} *derivative of* Es^{X_n} *becomes difficult as* m *increases. Thus it is important estimate moments* EX_n^m *from above. Inequalities for moments of critical and supercritical Galton-Watson processes were given first by S.V.Nagaev(see [5]). In his work mainly the analysis of generating functions was used. In the present work we provide estimations for moments of branching random processes with immigration. We use probability methods and well applied known inequalities for the sum of independent random variables. We consider Branching Random Processes with Immigration starting from random number of items. In this work we provide estimations from above for the moments of such processes.*
Keywords: *branching process, Galton-Watson process, immigration, moment, central moment, generating function*

О НЕРАВЕНСТВАХ ДЛЯ МОМЕНТОВ ВЕТВЯЩИХСЯ СЛУЧАЙНЫХ ПРОЦЕССОВ

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Аннотация: *оценки моментов случайных величин играют важную роль в теории вероятностей. Многие результаты были получены для моментов суммы независимых случайных величин (см. [4], [6]). Однако оценки моментов процессов вида (0.1) мало изучены. Хорошо известно, что моменты* $EX_n^m, m \geq 1$ *получаются путем дифференцирования в m раз производящей функции* Es^{X_n}*. Однако выражение для* m^{th} *производной от* Es^{X_n} *становится трудным по мере увеличения m. При этом важно оценить моменты* $EX_n^m m$ *сверху. Неравенства для моментов критических и сверхкритических процессов Гальтона-Ватсона впервые были даны С.В.Нагаевым (см. [5]). В его работе преимущественно использовался анализ производящих функций. В настоящей работе мы даем оценки моментов ветвящихся случайных процессов с иммиграцией. Мы используем вероятностные методы и хорошо применяемые известные неравенства для суммы независимых случайных величин.Мы рассматриваем ветвящиеся случайные процессы с иммиграцией, начиная со случайного числа элементов. В данной работе мы даем оценки сверху для моментов таких процессов.*
Ключевые слова*: ветвящийся процесс, процесс Гальтона-Ватсона, иммиграция, момент, центральный момент, производящая функция.*

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Introduction.

Let $\{\xi_{k,i}, \varepsilon_k : k, i \in N\}$ be independent, nonnegative, integer valued random variables such that $\{\xi_{k,i} : k, i \in N\}$ and $\{\varepsilon_k : k \in N\}$ are identically distributed.

Consider the branching processes with immigration $X_k, k \geq 0$ which is defined by the following recurrent relation:

$$X_0 = \eta,\ X_k = \sum_{j=1}^{X_{k-1}} \xi_{k,j} + \varepsilon_k,\ k \geq 1, \qquad (0.1)$$

where η is an nonnegative, integer valued random variable and independent of $\{\xi_{k,i}, \varepsilon_k : k, i \geq 1\}$ (see [1]). The sequence $\{X_k : k \in Z_+\}$ is called a branching process with immigration. We can interpret X_k as the size of the

k^{th} generation of a population in which the initial moments there are η particles, where $\xi_{k,j}$ is the number of offsprings of the j^{th} individual in the $(k-1)^{st}$ generation and ε_k is the number of immigrants contributing to the k^{th} generation.

The case when $\eta \equiv 1$ and $\varepsilon_k \equiv 0$ process (0.1) is fine known and thoroughly investigated process Galton-Watson (see, for example, [1],[2],[3])

Denote

$$m := E\xi_{1,1}, \qquad \sigma^2 := D\xi_{1,1}, \qquad \gamma_p := E\xi_{1,1}^p, \theta_p := E|\xi_{1,1} - m|^p,$$

$$\lambda := E\varepsilon_1, \qquad b^2 := D\varepsilon_1, \qquad \tau_p := E\varepsilon_1^p, \qquad \delta_p := E|\varepsilon_1 - \lambda|^p,$$

$$\nu_p := E\eta^p, \qquad s^2 := D\eta, \qquad \beta_p := E|\eta - E\eta|^p$$

Assume here and in what follows that all moments are finite.

The cases $m < 1, m = 1, m > 1$ are referred to respectively as subcritical, critical and supercritical.

Estimations for moments of random variables play an important role in the probability theory. Many results were obtains for the moments of the sum of independent random variables(see [4],[6]). However, the estimates for moments of processes of the form (0.1) have been little studied. It is well known that moments $EX_n^m, m \geq 1$, are obtained by m times differentiating the generating function Es^{X_n}. However, the expression for m^{th} derivative of Es^{X_n} becomes difficult as m increases. Thus it is important estimate moments EX_n^m from above. Inequalities for moments of critical and supercritical Galton-Watson processes were given first by S.V.Nagaev(see [5]). In his work mainly the analysis of generating functions was used.

In the present work we provide estimations for moments of branching random processes with immigration. We use probability methods and well applied known inequalities for the sum of independent random variables.

Main results

Assume we are given a process (0.1). In the next two theorems we provide estimations for the moments and central moments of X_n

Theorem 2.1 *We have the following inequalities:*

1. for $0 < p \leq 1$ and $m \neq 1$
$$EX_n^p \leq \left(\nu_1 m^{n-1} + \frac{m^{n-1} - 1}{m-1}\lambda\right)\gamma_p + \tau_p;$$
2. for $0 < p \leq 1$ and $m = 1$
$$EX_n^p \leq (\nu_1 + \lambda n)\gamma_p + \tau_p;$$
3. for $p > 1$ and $2^{p-1}\gamma_p \neq 1$
$$EX_n^p \leq \left(2^{p-1}\gamma_p\right)^n \nu_p + \frac{2^{p-1}\tau_p\left[\left(2^{p-1}\gamma_p\right)^n - 1\right]}{2^{p-1}\gamma_p - 1}$$
4. for $p > 1$ and $2^{p-1}\gamma_p = 1$
$$EX_n^p \leq \nu_p + 2^{p-1}\tau_p n$$

Theorem 2.2 *We have the following inequalities:*

1. for $0 < p \leq 1$ and $m \neq 1$
$$E|X_n - EX_n|^p \leq m^{n-1}\theta_p\left(\nu_1 + \frac{\lambda}{m-1}\right)\frac{\left(m^{(p-1)n} - 1\right)}{m^{(p-1)} - 1} +$$
$$+\left(\delta_p - \frac{\lambda\theta_p}{m-1}\right)\frac{(m^{np} - 1)}{m^p - 1} + m^{np}\beta_p$$
2. for $0 < p \leq 1$ and $m = 1$
$$E|X_n - EX_n|^p \leq (\theta_p\nu_1 + \delta_p)n + \tfrac{(n-1)n}{2}\theta_p\lambda + \beta_p$$
3. for $1 < p \leq 2$ and $m \neq 1$
$$E|X_n - EX_n|^p \leq 2^{2(p-1)}n^{p-1}[2\theta_p m^{n-1}\left(\nu_1 + \frac{\lambda}{m-1}\right)\frac{m^{n(p-1)} - 1}{m^{p-1} - 1} +$$
$$+\left(\delta_p - \frac{2\theta_p\lambda}{m-1}\right)\frac{m^{pn} - 1}{m^p - 1}] + 2^{p-1}m^{np}\beta_p$$
4. for $1 < p \leq 2$ and $m = 1$
$$E|X_n - EX_n|^p \leq 2^{2(p-1)}n^p[2\theta_p\nu_1 + \delta_p + (n-1)\theta_p\lambda] + 2^{p-1}\beta_p$$
5. for $p > 2$ and $m \neq 1,\ m \neq (2^{\frac{p}{2}-1}\gamma_{\frac{p}{2}})^{\frac{1}{p}}$

$$E|X_n - EX_n|^p \leq 2^{2(p-1)} n^{p-1} [C(p)\theta_p \left(\nu_{\frac{p}{2}} + \frac{2^{\frac{p}{2}-1}\tau_{\frac{p}{2}}}{2^{\frac{p}{2}-1}\gamma_{\frac{p}{2}} - 1} \right) \frac{m^{pn} - (2^{\frac{p}{2}-1}\gamma_{\frac{p}{2}})^n}{m^p - 2^{\frac{p}{2}-1}\gamma_{\frac{p}{2}}} +$$

$$+ \left(\delta_p - \frac{2^{\frac{p}{2}-1} C(p)\theta_p \tau_{\frac{p}{2}}}{2^{\frac{p}{2}-1}\gamma_{\frac{p}{2}} - 1} \right) \frac{m^{np} - 1}{m^p - 1}] + 2^{p-1} m^{np} \beta_p$$

where $C(p)$ is a constant depending only on p.

6. for $p > 2$ and $m = 1,\ 2^{\frac{p}{2}-1}\gamma_{\frac{p}{2}} = 1$

$$E|X_n - EX_n|^p \leq 2^{2(p-1)} n^p [C(p)\theta_p \nu_{\frac{p}{2}} + \delta_p + 2^{\frac{p}{2}-1} C(p)\theta_p \tau_{\frac{p}{2}}(n-1)] + 2^{p-1}\beta_p$$

where $C(p)$ is a constant depending only on p.

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